Calculus 140, section 5.7 The Logarithm

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In Algebra/Precalculus classes, you were handed (on a silver platter, as it were) items of information that had to wait until Calculus for a formal proof.

We're now faced with much the same situation with regard to the function $y = \ln x$ and its properties, which we have encountered several times in our exploration of Calculus so far.

We begin the formal treatment with the rational function $f(x) = \frac{1}{x}$ which is continuous on $(0, \infty)$.

Next we define a function $G(x) = \int_1^x \frac{1}{t}\,dt$ for all $x > 0$. Note, in particular, that $G(1) = \int_1^1 \frac{1}{t}\,dt = 0$.

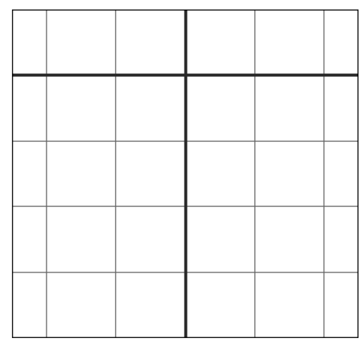
By Theorem 5.12 (section 5.4) G is differentiable on $(0, \infty)$, and $\frac{dG}{dx} = \frac{1}{x}$.

Since we also have $\ln(1) = 0$ and $\frac{d}{dx}[\ln x] = \frac{1}{x}$, by Theorem 4.6 (which implies uniqueness) we can define

$$\ln(x) = \int_1^x \frac{1}{t}\,dt .$$

See the text for development of the graph of $y = \ln x$, including limits, and the connection to Euler's number e.

Example A: Given $f(x) = \ln(1 - x^2)$ find the domain, intercepts, relative extreme values, inflection points, concavity and asymptotes, then draw the graph.



One more important “silver platter” item that we can now prove.

Fix a number $b > 0$. Then for $x > 0$, let $g(x) = \ln(bx)$. Next, $g'(x) = \frac{1}{bx} * \frac{d}{dx}(bx) = \frac{1}{bx} * b = \frac{1}{x}$ [Chain Rule].

Since $\frac{d}{dx}[g(x)] = \frac{1}{x} = \frac{d}{dx}[\ln x]$, then by Theorem 4.6 (section 4.3) we can state that $g(x) = \ln(bx) = \ln(x) + C$.
For $x = 1$, we get $g(1) = \ln(b) = \ln(1) + C \quad \Rightarrow \quad \ln(bx) = \ln(b) + \ln(x)$ for all x.

Theorem 5.21: “For all $b > 0$ and $c > 0$, ln bc = ln b + ln c.”

This is the **Law of Logarithms** introduced in Algebra/Precalculus and re-introduced in the text in Chapter 1. Your text notes that the other properties of logarithms can be easily derived from the Law of Logarithms.

The natural logarithm function, ln(x), can be used in a process called *logarithmic differentiation* to ease the differentiation of products and quotients involving multiple terms. Note that for any function $g(x) = \ln[f(x)]$, by the chain rule $g'(x) = \frac{1}{f(x)} * f'(x) = \frac{f'(x)}{f(x)}$.

Example B: Given the polynomial $g(x) = (x+3)(x+1)^2(x-1)^3$, find the first derivative.

Using logarithmic differentiation,

(a) Take the natural logarithm of both sides and use logarithm properties to expand:

$$\ln[g(x)] = \ln\left[(x+3)(x+1)^2(x-1)^3\right] = \ln(x+3) + 2\ln(x+1) + 3\ln(x-1)$$

(b) Take the derivative of ln [g(x)]: $\frac{g'(x)}{g(x)} = \frac{1}{x+3} + \frac{2}{x+1} + \frac{3}{x-1}$

(c) Solve algebraically for $g'(x)$: $g'(x) = \left[\frac{1}{x+3} + \frac{2}{x+1} + \frac{3}{x-1}\right] * g(x)$

(d) Back-substitute for g(x): $g'(x) = \left[\frac{1}{x+3} + \frac{2}{x+1} + \frac{3}{x-1}\right] * \left[(x+3)(x+1)^2(x-1)^3\right]$

Example C: Use logarithmic differentiation to find the first derivative of $h(x) = \frac{(x^3-2)(x^2-3)^4}{\sqrt{8x-5}}$.

answer: $\left[\frac{3x^2}{x^3-2} + \frac{8x}{x^2-3} - \frac{4}{8x-5}\right] * \frac{(x^3-2)(x^2-3)^4}{\sqrt{8x-5}}$

Now, we consider $g(x) = \ln(-x)$, with domain $(-\infty, 0)$.

Using the Chain Rule, $g'(x) = \frac{1}{-x} * \frac{d}{dx}(-x) = \frac{1}{-x} * (-1) = \frac{1}{x}$. Recall also that $\frac{1}{x} = \frac{d}{dx}(\ln x)$.

We conclude that the function $\ln|x|$, which equals $\ln(-x)$ on $(-\infty, 0)$ and equals $\ln(x)$ on $(0, \infty)$, is an antiderivative of $y = \frac{1}{x}$ on its entire domain, $(-\infty, 0)$ union $(0, \infty)$.

As a result, when we integrate $\frac{1}{x}$, we no longer need to limit ourselves to domains of positive values as we did in Lecture 5.5 Example H and Lecture 5.6 Example D. We can now state $\int \frac{1}{x}\, dx = \ln|x| + C$ for all $x \neq 0$.

Examples D: Evaluate $\int \frac{5}{x}\, dx$ and $\int \frac{1}{6x}\, dx$. *answers*: $5\ln|x| + C, \frac{1}{6}\ln|x| + C$

Example E: Evaluate $\int \left(4(1-2x)^{-1}\right) dx$. *answer*: $-2\ln|1-2x| + C$

Example F: Evaluate $\int_{-2}^{1} \frac{x^2}{x^3 - 8}\, dx$. *answer*: $\frac{1}{3}\ln\left(\frac{7}{16}\right)$

Example G: Evaluate $\int \tan x\, dx$. *answer*: $-\ln|\cos x| + C = \ln|\sec x| + C$